Positive entries of stable matrices

Shmuel Friedland
University of Illinois at Chicago
Chicago, USA

Daniel Hershkowitz,
Department of Mathematics
Technion, Israel Institute of Technology
Haifa, Israel

Siegfried M. Rump
Inst. f. Computer Science III
Technical University Hamburg-Harburg Schwarzenbergstr.
95 21071 Hamburg Germany

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Abstract

We show that any real stable matrix matrix of order greater than 1 has at least two positive entries and this result is sharp.

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1 Introduction

Let $M_n(\mathbb{R})$ be the algebra of $n \times n$ real valued matrices. For $A = (a_{ij}) \in M_n(\mathbb{R})$ let $\lambda(A) := \{\lambda_1(A), ..., \lambda_n(A)\} \subset \mathbb{C}$ be the set of complex eigenvalues of $A$ listed with their multiplicities. Recall that $A$ is called stable if all the eigenvalues of $A$ have positive real parts. Assume that $A$ is stable. It is quite straightforward to show that the following dichotomy holds. Either all diagonal elements of $A$ are positive or $A$ must have at least two positive entries.
entries: one on the main diagonal and one on the off-diagonal, see Lemma
2.4. The aim of this note to show that for any set of \( n > 1 \) complex numbers
\( \zeta := \{z_1, \ldots, z_n\} \subset \mathbb{C} \), such that \( \bar{\zeta} = \zeta \) and \( \Re z_i > 0 \), \( i = 1, \ldots, n \) there exists a stable matrix \( A \) with exactly two positive entries such that \( \lambda(A) = \zeta \).

2 Main results

Lemma 2.1 Let \( A = (a_{ij})_1^n \in M_n(\mathbb{R}) \) be a stable matrix. Then either
all the diagonal elements of \( A \) are positive or \( A \) has at least one positive
diagonal element and one positive off-diagonal element.

Proof. First note that \( \sum_{i=1}^n = \text{tr} A = \sum_{i=1}^n \lambda_i(A) > 0 \). Hence at least
one diagonal element of \( A \) is positive. Assume that that all off-diagonal
elements of \( A \) are nonpositive, i.e. \( A \) is a \( Z \)-matrix. Since
\( A \) is stable it follows that \( A \) is a nonsingular \( M \)-matrix [1], hence all the diagonal entries
of \( A \) are positive. \( \square \)

Let \( \zeta := \{z_1, \ldots, z_n\} \subset \mathbb{C} \). Denote by \( e_1(\zeta), \ldots, e_n(\zeta) \) the \( n \) elementary
symmetric polynomials \( (esp) \) of \( \zeta \):

\[
e_k(\zeta) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} z_{i_1} \cdots z_{i_k}, \quad k = 1, \ldots, n.
\]

\( \zeta \) is called symmetric, with respect to the \( X \) axis, if \( \bar{\zeta} = \zeta \) if and only if \( e_1(\zeta), \ldots, e_n(\zeta) \in \mathbb{R} \). We say that \( \zeta \) has positive
esp if \( e_k(\zeta) \in (0, \infty) \) for \( k = 1, \ldots, n \). \( \zeta \) is called a stable
set if \( \Re z_i > 0 \) for \( i = 1, \ldots, n \). The following result is well known and we bring its short proof
for completeness.

Proposition 2.2 Let \( \zeta = \{z_1, \ldots, z_n\} \subset \mathbb{C} \), be a symmetric stable set. Then \( \zeta \) has positive esp.

Proof. By induction on \( n \). For \( n = 1, 2 \) the result is trivial. Assume
that the result holds for any \( n \leq m \) where \( m \geq 2 \). Let \( n = m + 1 \). Clearly
\( e_1(\zeta) > 0 \). Let \( k \in [2, m + 1] \cap \mathbb{Z} \). Assume first that \( z_{m+1} \in (0, \infty) \).
Let \( \zeta' := \{z_1, \ldots, z_m\} \). Then \( e_k(\zeta) = e_k(\zeta') + z_{m+1}e_{k-1}(\zeta') > 0 \). Assume
now that \( z_{m+1} = \tilde{z}_m \in \mathbb{C} \setminus \mathbb{R} \). Let \( \zeta' := \{z_1, \ldots, \tilde{z}_{m-1}\} \). Then \( e_k(\zeta) = e_k(\zeta') + (z_m + z_{m+1})e_{k-1}(\zeta') + z_m\tilde{z}_{m+1}e_{k-2}(\zeta') > 0 \), where \( e_k(\zeta') = 0 \) for
\( k > m - 1 \) and \( e_0(\zeta') = 1 \). \( \square \)
Let $\zeta = \{z_1, \ldots, z_n\} \subset \mathbb{C}$ be a symmetric set. Then $\zeta$ has positive esp if and only if $p(x) := \prod_{i=1}^{n}(x + z_i)$ has positive coefficients. Hence if $\zeta$ has positive esp then $\zeta \cap \mathbb{R} \subset (0, \infty)$, i.e. any real $z \in \zeta$ is a positive number.

Clearly for $n = 1, 2$ $\zeta$ is stable if and only if $\zeta$ has positive esp. For $n \geq 3$ it is not difficult to show there exists a nonstable $\zeta$ with positive esp. (It is enough to show that this statement holds for $n = 3$, e.g. $\zeta = \{-1 + 3\sqrt{-1}, -1 - 3\sqrt{-1}, 3\}$.)

Let $A \in M_n(\mathbb{R})$. Then $e_k(\lambda(A))$ is the sum of all $k \times k$ principal minors.

We say that the eigenvalues of $A$ have positive esp if $\lambda(A)$ have positive esp.

**Corollary 2.3** Let $A \in M_n(\mathbb{R})$ be a stable matrix. Then the eigenvalues of $A$ have positive esp.

The proof of Lemma 2.4 and one of the many conditions for $Z$-matrix to be a nonsingular $M$-matrix [1] yield

**Lemma 2.4** Let $A = (a_{ij})_{1}^{n} \in M_n(\mathbb{R})$ be a matrix whose sum of all $k \times k$ principle minors are positive for $k = 1, \ldots, n$. Then either all the diagonal elements of $A$ are positive or $A$ has at least one positive diagonal element and one positive off-diagonal element.

Recall that $A \in M_n(\mathbb{C})$ is called nonderogatory if for each eigenvalue $\lambda \in \lambda(A)$ the Jordan canonical form of $A$ has exactly one Jordan block corresponding to $A$. Equivalently, the minimal polynomial of $A$ is equal to the characteristic polynomial of $A$.

**Lemma 2.5** Let $1 < n \in \mathbb{N}$ and $\zeta = \{\zeta_1, \ldots, \zeta_n\} \subset \mathbb{C}$. Then the following three matrices are diagonally similar, are nonderogatory and with the eigenvalue set equal to $\zeta$:

$$C_1(\zeta) := \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-1}e_n(\zeta) \\
1 & 0 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-2}e_{n-1}(\zeta) \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-3}e_{n-2}(\zeta) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & -e_2(\zeta) \\
0 & 0 & 0 & \ldots & 0 & 0 & 1 & e_1(\zeta)
\end{pmatrix},$$

$$C_2(\zeta) := \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 & e_n(\zeta) \\
-1 & 0 & 0 & \ldots & 0 & 0 & 0 & e_{n-1}(\zeta) \\
0 & -1 & 0 & \ldots & 0 & 0 & 0 & e_{n-2}(\zeta) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -1 & 0 & e_2(\zeta) \\
0 & 0 & 0 & \ldots & 0 & 0 & -1 & e_1(\zeta)
\end{pmatrix}.$$
\( C_3(\zeta) := \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 & 0 & -e_n(\zeta) \\
-1 & 0 & 0 & \ldots & 0 & 0 & 0 & -e_{n-1}(\zeta) \\
0 & -1 & 0 & \ldots & 0 & 0 & 0 & -e_{n-2}(\zeta) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & -1 & 0 & -e_2(\zeta) \\
0 & 0 & 0 & \ldots & 0 & 0 & 1 & e_1(\zeta)
\end{pmatrix}. \)

**Proof.** The matrix \( C_1(\zeta) \) is the companion matrix of the polynomial 
\( q(x) = \prod_{i=1}^{n}(x - z_i) \). Hence \( \lambda(C_1(\zeta)) = \zeta \) and \( C_1(\zeta) \) is nonderogatory. Clearly

\[
C_2(\zeta) = D_1 C_1(\zeta) D_1, \quad D_1 = \text{diag}((-1)^1, (-1)^2, \ldots, (-1)^n),
C_2(\zeta) = D_2 C_2(\zeta) D_2, \quad D_2 = \text{diag}(1, 1, \ldots, 1, -1).
\]

**Theorem 2.6** Let \( n > 1 \) and \( \zeta = \{\zeta_1, \ldots, \zeta_n\} \subset \mathbb{C} \) be a symmetric set. If \( \zeta \) has positive esp then there exists \( A \in M_n(\mathbb{R}) \) such that \( \lambda(A) = \zeta \) and \( A \) has one positive diagonal entry and one positive off-diagonal entry, while all other entries are nonpositive. In particular, any nonderogatory stable matrix \( A \in M_n(\mathbb{R}) \) is similar to \( B \in M_n(\mathbb{R}) \) which has exactly two positive entries.

### 3 Additional results

Here we can put the following stuff.

1. Other examples of stable matrices with two positive entries, as Siegfried suggested in his last e-mail.
2. By perturbing \( C_3(\zeta) \) we can change all zero entries to negative ones. So we have examples with no zeros.
3. Can we get better results by perturbations using similarity, i.e. without changing the spectrum of \( C_3(\zeta) \).

That is consider \( (I + X)C_3(I - X) = C_3 + XC_3 - C_3X + \text{higher order.} \)

Under what conditions it is possible to find real matrix \( X \) such that \( XC_3 - C_3X \) have positive entries where \( C_3 \) has zero entries. (Linear programming problem, use the dual formulation.)

**References**