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OPTIMAL SCALING FOR P-NORMS AND COMPONENTWISE DISTANCE TO SINGULARITY

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Abstract. In this note we give lower and upper bounds for the optimal p-norm condition number achievable by two-sided diagonal scalings. There are no assumptions on the irreducibility of certain matrices. The bounds are shown to be optimal for the 2-norm. For the 1-norm and inf-norm the (known) exact value of the optimal condition number is confirmed. We give means how to calculate the minimizing diagonal matrices. Furthermore, a class of new lower bounds for the componentwise distance to the nearest singular matrix is given. They are shown to be superior to existing ones.

1. Introduction and notation. Through the paper let A be an $n \times n$ nonsingular real matrix. The condition number of A is defined by

$$\kappa_p(A) := \lim_{\varepsilon\to 0} \sup_{\|\Delta A\|_p \le \varepsilon \|A\|_p} \left(\frac{\|(A+\Delta A)^{-1} - A^{-1}\|_p}{\varepsilon \|A^{-1}\|_p} \right),$$

where $\|\cdot\|_p$ denotes the Hölder p-norm, $1 \le p \le \infty$. It is well known (cf. [4, Theorem 6.4,7.2]) that

$$\begin{aligned} \kappa_p(A) &= \lim_{\varepsilon\to 0} \sup \left\{ \frac{\|\Delta x\|_p}{\varepsilon \|x\|_p} : Ax = b = (A+\Delta A)(x+\Delta x), \|\Delta A\|_p \le \varepsilon \|A\|_p \right\} \\ &= \|A^{-1}\|_p \|A\|_p. \end{aligned}$$

The question arises what is the minimum condition number achievable by two-sided diagonal scaling, i.e. the value of

$$\mu_p := \inf_{D_1, D_2 \in \mathcal{D}_n} \kappa_p(D_1 A D_2), \tag{1}$$

where $\mathcal{D}_n \subseteq M_n(\mathrm{I\!R})$ denotes the set of $n \times n$ diagonal matrices with positive diagonal elements. Various results are known (cf. [4, Section 7]). Especially for the ∞-norm Bauer [2] shows (cf. [4, Theorem 7.8, Exercise 7.9])[1]

$$\mu_\infty = \varrho(|A^{-1}||A|) \quad \text{if } |A^{-1}||A| \text{ and } |A||A^{-1}| \text{ are irreducible,} \tag{2}$$

ϱ denoting the spectral radius. In the following we derive general two sided bounds for μ_p including (2) without irreducibility condition. We will prove

$$\mu_p \le \varrho(|A^{-1}||A|) \le n^{2\min(1/p, 1-1/p)} \cdot \mu_p. \tag{3}$$

The bounds differ at most by a factor n (for the 2-norm). For $p \in \{1, \infty\}$ the exact value of μ_p is $\varrho(|A^{-1}||A|)$ for general A, and we show that the bounds in (3) are sharp for the 2-norm. Furthermore, the minimizing diagonal matrices can be calculated explicitly.

Finally, the results are used to derive new lower bounds for the componentwise distance to the nearest singular matrix. The bounds are shown to be superior to existing ones. Frequently, the derived methods allow to calculate the exact value of the componentwise distance to the nearest singular matrix. This seems remarkable because Polak and Rohn [7] showed that the computation of this number is NP-hard. Computational results of matrices up to dimension $n = 50$ are presented.

Throughout the paper we will use absolute value and comparison of matrices always entrywise. For example, $|\widetilde{E}| \le E$ is equivalent to $|\widetilde{E}_{ij}| \le E_{ij}$ for all i, j.

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[1] The starting point of this paper was the question by N. Higham whether the irreduciblility assumptions in [4, Theorem 7.8] can be omitted.

2. A result from matrix theory. In this section we will prove the following theorem.

THEOREM 2.1. *Let $A_1, \ldots, A_k \in M_n(\mathbb{R})$ be given and denote by m, $0 \le m \le k$, the number of nonnegative matrices among the A_ν. For fixed $p, 1 \le p \le \infty$, define*

$$\mu := \inf_{D_1,\ldots,D_k \in \mathcal{D}_n} \|D_1^{-1} A_1 D_2\|_p \cdot \|D_2^{-1} A_2 D_3\|_p \cdot \ldots \cdot \|D_k^{-1} A_k D_1\|_p. \tag{4}$$

Then for $\alpha := n^{\min(1/p,1-1/p)}$ we have

$$\mu \le \varrho(|A_1||A_2| \ldots |A_k|) \le \alpha^{k-m} \mu. \tag{5}$$

The inequalities are equalities if all A_ν are nonnegative. The left bound is sharp for all p and all n. For $p = 2$, the right bound is sharp at least for the infinitely many values of n, where a Hadamard matrix exists.

Before we prove Theorem 2.1 we need some auxiliary results. First we generalize a result by Albrecht [1] to arbitrary nonnegative matrices.

LEMMA 2.2. *Let nonnegative $M \in M_n(\mathbb{R})$ be given. Then for all p, $1 \le p \le \infty$,*

$$\inf_{D \in \mathcal{D}} \|D^{-1} M D\|_p = \varrho(M).$$

PROOF. Albrecht [1] proved this result for irreducible M. Furthermore, he proved the infimum to be a minimum in that case and gave explicit formulas for the minimizing D depending on the left and right Perron vector of M. For general M, we may assume without loss of generality M to be in its irreducible normal form

$$M = \begin{pmatrix} M_{11} & \ldots & M_{1k} \\ & \ddots & \vdots \\ 0 & & M_{kk} \end{pmatrix}, \tag{6}$$

where the $M_{\nu\nu}$, $1 \le \nu \le k$ denote an irreducible or a 1×1 zero matrix. This is because (6) is achieved by a permutational similarity transform $P^T M P$ and $\|D^{-1} P^T M P D\|_p = \|(PDP^T)^{-1} M PDP^T\|_p$ with diagonal PDP^T. By Albrecht's theorem there are $D_{\nu\nu} \in \mathcal{D}$ of appropriate size such that

$$\|D_\nu^{-1} M_{\nu\nu} D_\nu\|_p = \varrho(M_{\nu\nu}).$$

Set $D = \operatorname{diag}(D_1, \varepsilon D_2, \ldots, \varepsilon^{k-1} D_k) \in \mathcal{D}_n$. Then

$$D^{-1} M D = \begin{pmatrix} D_1^{-1} M_{11} D_1 & & \mathcal{O}(\varepsilon) \\ & \ddots & \\ 0 & & D_k^{-1} M_{kk} D_k \end{pmatrix},$$

a block upper triangular matrix with $\mathcal{O}(\varepsilon)$ terms above the block diagonal. Therefore

$$\begin{aligned} \|D^{-1} M D\|_p &= \max_\nu \|D_\nu^{-1} M_{\nu\nu} D_\nu\|_p + \mathcal{O}(\varepsilon) \\ &= \max_\nu \varrho(M_{\nu\nu}) + \mathcal{O}(\varepsilon) = \varrho(M) + \mathcal{O}(\varepsilon). \end{aligned} \tag{7}$$

■

Using Albrecht's result a D with (7) can be constructed explicitly for every $\varepsilon > 0$. For the 2-norm and irreducible nonnegative M this is particularly easy. Denote $r := \varrho(M) > 0$, $Mx = rx > 0$, $M^T y = ry > 0$ and define $D \ge 0$ such that $D^{-1} x = Dy$. Then for $N := D^{-1} M D$,

$$\begin{aligned} N^T N \cdot Dy &= N^T D^{-1} M D \cdot D^{-1} x &= r N^T D^{-1} x \\ &= r D M^T D^{-1} \cdot Dy &= r^2 Dy, \end{aligned}$$

showing $Dy > 0$ to be the Perron vector of $N^T N \ge 0$, and therefore $\|D^{-1} M D\|_2 = r$.

For the proof of Theorem 2.1 and general M we also need

$$\|M\|_p \leq \| \, |M| \, \|_p \leq n^{\min(1/p, 1-1/p)} \cdot \|M\|_p, \tag{8}$$

which is true for all $M \in M_n(\mathbb{R})$ and all $1 \leq p \leq \infty$ (cf. [4, exercise 6.15]).

Proof of Theorem 2.1. First we transform the definition of μ into the minimization of the p-norm of a certain matrix. Define the block cyclic matrix

$$A = \begin{pmatrix} 0 & A_1 & & & \\ & 0 & A_2 & & \\ & & \dots & & \\ & & & & A_{k-1} \\ A_k & & & & 0 \end{pmatrix} \in M_{kn}(\mathbb{R}). \tag{9}$$

Then $\|A\|_p = \max\limits_{1 \leq \nu \leq k} \|A_\nu\|_p$ because the p-norm is invariant under row or column permutations. For $D := \operatorname{diag}(D_1, \dots, D_n) \in \mathcal{D}_{kn}$ it follows

$$\|D^{-1}AD\|_p = \max_{1 \leq \nu \leq k} \|D_\nu^{-1} A_\nu D_{\nu+1}\| \quad \text{with } k+1 \text{ interpreted as } 1. \tag{10}$$

This and the infimum in (4) imply

$$\mu^{1/k} \leq \inf_{D \in \mathcal{D}} \|D^{-1}AD\|_p. \tag{11}$$

Now we can prove the inequalities in (5). Suppose, according to Lemma 2.2, $\widetilde{D} \in \mathcal{D}_{kn}$ is given such that $\varrho(|A|) + \varepsilon = \|\widetilde{D}^{-1}|A|\widetilde{D}\|_p$. Then by (11) and (10),

$$\mu^{1/k} \leq \|\widetilde{D}^{-1}A\widetilde{D}\|_p \leq \|\widetilde{D}^{-1}|A|\widetilde{D}\|_p = \varrho(|A|) + \varepsilon.$$

But the cyclicity of A defined by (9) implies $\varrho(|A|) = \varrho(|A_1| \dots |A_k|)$, and this proves the left inequality in (4). For the right inequality suppose $\widetilde{D}_1, \dots, \widetilde{D}_k \in \mathcal{D}_n$ be given such that

$$\mu + \varepsilon = \|\widetilde{D}_1^{-1} A_1 \widetilde{D}_2\|_p \cdot \ldots \cdot \|\widetilde{D}_k^{-1} A_k \widetilde{D}_1\|_p.$$

Then by (8),

$$\begin{aligned} \mu + \varepsilon &\geq \alpha^{-(k-m)} \cdot \|\widetilde{D}_1^{-1}|A_1|\widetilde{D}_2\|_p \cdot \ldots \cdot \|\widetilde{D}_k^{-1}|A_k|\widetilde{D}_1\|_p \\ &\geq \alpha^{-(k-m)} \cdot \|\widetilde{D}_1^{-1}|A_1| \cdot |A_2| \cdot \ldots \cdot |A_k|\widetilde{D}_1\|_p \\ &\geq \alpha^{-(k-m)} \cdot \varrho(\widetilde{D}_1^{-1}|A_1| \cdot \ldots \cdot |A_k|\widetilde{D}_1) \\ &= \alpha^{-(k-m)} \cdot \varrho(|A_1| \cdot \ldots \cdot |A_k|). \end{aligned}$$

The left inequality in (5) is obviously sharp for all A_ν equal to the identity matrix I. Let $p = 2$ and let H be an $n \times n$ Hadamard matrix, i.e. $|H_{ij}| = 1$ for all i, j, and $H^T H = nI$. Define $A_\nu = n^{-1/2} H$ for $1 \leq \nu \leq n$. Then the A_ν are orthogonal, and obviously $\mu = 1$. On the other hand, $\varrho(|A_1| \cdot \ldots \cdot |A_n|)^k = n^{k/2} = \alpha^k$, such that the right inequality in (5) is sharp for $p = 2$ and infinitely many values of n. The theorem is proved. ■

Note that there is in fact equality in (11) although this is, as remarked by a referee, not necessary for the proof. This is because in optimality all norms $\|D_\nu^{-1} A_\nu D_{\nu+1}\|$ must be equal, and to see this it suffices to use suitable multiples $D_\nu := \alpha_\nu I$ of the identity matrix.

Note that for $A_1 = \ldots = A_k = Q$ being some orthogonal matrix, e.g. `orth(rand(n))` in Matlab [5] notation, freqently $|Q_{ij}|$ is of the order $n^{1/2}$ for all i, j, and $\varrho(|Q|)$ is of the order $\sqrt{n}$. This means that for general orthogonal $Q = A_1 = \ldots = A_k$ the right bound (5) is almost sharp.

3. Optimal two-sided scaling. With these preparations we can state our two-sided bounds for the optimal p-condition number with respect to two-sided diagonal scaling.

THEOREM 3.1. *Let nonsingular $A \in M_n(\mathrm{I\!R})$ be given and let $1 \le p \le \infty$. Define*

$$\mu_p := \inf_{D_1, D_2 \in \mathcal{D}_n} \kappa_p(D_1 A D_2).$$

Then

$$\mu_p \le \varrho(|A^{-1}|\,|A|) \le \alpha_p^2 \cdot \mu_p$$

with $\alpha_p := n^{\min(1/p, 1-1/p)}$. For $p \in \{1, \infty\}$ both inequalities are equalities. The left bound is sharp for all p. For $p = 2$ the right inequality is sharp at least for the infinitely many values of n where an $n \times n$ Hadamard matrix exists.

The proof is an immediate consequence of Theorem 2.1. Note that by the proof of Lemma 2.2 one may find D_1, D_2 explicitly such that $\kappa_p(D_1 A D_2)$ approximates $\varrho(|A^{-1}|\,|A|)$, a value not too far from the optimum.

For the computation of the true value of the minimum 2-norm condition number we mention the following. By (10) and (11) we may transform the problem into

$$\inf_{D_1, D_2 \in \mathcal{D}_n} \kappa_2(D_1 A D_2) = \inf_{D \in \mathcal{D}_{2n}} \|D^{-1} \begin{pmatrix} 0 & A \\ A^{-1} & 0 \end{pmatrix} D\|_2^2. \tag{12}$$

In 1990, Sezginer and Overton [9] gave an ingenious proof for the fact that for fixed $M \in M_n(\mathrm{I\!R})$, the function $\|e^{-D} M e^{D}\|_2$ is convex in the $D_{\nu\nu}$. This offers reasonable ways to compute μ_2 by means of convex optimization (see also [10]).

4. Componentwise distance to singularity. In [3] the componentwise distance to the nearest singular matrix $\underline{\omega}(A, E)$ was investigated. It is defined by

$$\underline{\omega}(A, E) := \min\{\alpha : |\widetilde{E}| \le \alpha E,\ \det(A + \widetilde{E}) = 0\}. \tag{13}$$

As before we will assume A to be nonsingular. A well known lower bound is

$$\varrho(|A^{-1}|E)^{-1} \le \underline{\omega}(A, E). \tag{14}$$

This follows easily by

$$\det(A + \widetilde{E}) = 0 = \det(I + A^{-1}\widetilde{E}),\ |\widetilde{E}| \le \alpha |E|,\ \alpha = \underline{\omega}(A, E) \quad \Rightarrow$$
$$1 \le \varrho(A^{-1}\widetilde{E}) \le \varrho(|A^{-1}|\,|\widetilde{E}|) \le \alpha \cdot \varrho(|A^{-1}|E),$$

the latter deductions using classical Perron-Frobenius Theory.

In interval analysis this is a well known tool to show nonsingularity of an interval matrix $\mathcal{A} := [A - E, A + E] = [\widetilde{A} :\ A - E \le \widetilde{A} \le A + E]$. An interval matrix is called nonsingular, if all $\widetilde{A} \in \mathcal{A}$ share this property. Now $\varrho(|A^{-1}|E) < 1$ implies $\mathcal{A}$ to be nonsingular, and therefore an interval matrix with this property is called strongly regular [6].

The lower bound (14) was for a long time the only known (simple) criterion for regularity of an interval matrix. In [8, Corollary 1.10], we proved

$$\left[\|A^{-1}\|_2 \|E\|_2\right]^{-1} \le \underline{\omega}(A, E). \tag{15}$$

It was shown that for both criteria (14) and (15) there are examples where the one is satisfied and the other is not.

Our analysis yields new lower bounds for $\underline{\omega}(A, E)$ and henceforth new sufficient criteria for nonsingularity of an interval matrix. The new lower bounds will be shown to be superior to the lower bound $\varrho(|A^{-1}|E)^{-1}$.

Note that a lower bond $r \le \underline{\omega}(A,E)$ implies *all* matrices $\widetilde{A}$ with $|\widetilde{A} - A| < rE$ to be nonsingular. Also note that the computation of $\underline{\omega}(A,E)$ is known to be NP-hard [7].

The reciprocal of $\|A^{-1}\|_p \|E\|_p$ is always a lower bound for $\underline{\omega}(A,E)$. This is seen as before by

$$\det(A+\widetilde{E}) = 0 = \det(I + A^{-1}\widetilde{E}),\ |\widetilde{E}| \le \alpha E,\ \alpha = \underline{\omega}(A,E) \quad \Rightarrow$$
$$1 \le \varrho(A^{-1}\widetilde{E}) \le \|A^{-1}\|_p \|\widetilde{E}\|_p \le \|A^{-1}\|_p \|\, |\widetilde{E}|\, \|_p \le \alpha \cdot \|A^{-1}\|_p\ \|E\|_p$$

with the aid of (8). But $\underline{\omega}(A,E)$ is invariant under left and right diagonal scaling, so

$$\forall\, D_1, D_2 \in \mathcal{D}_n :\ \left[\|D_2^{-1}A^{-1}D_1^{-1}\|_p \|D_1 E D_2\|_p\right]^{-1} \le \underline{\omega}(A,E).$$

The infimum of the left hand side can be bounded by means of Theorem 2.1) and we obtain the following.

THEOREM 4.1. *Let nonsingular $A \in M_n(\mathrm{IR})$ and $0 \le E \in M_n(\mathrm{IR})$ be given. Define for $1 \le p \le \infty$*

$$\nu_p := \inf_{D_1, D_2 \in \mathcal{D}_n} \|D_2^{-1}A^{-1}D_1^{-1}\|_p \|D_1 E D_2\|_p. \tag{16}$$

Then

$$\nu_p^{-1} \le \underline{\omega}(A,E) \textit{ for all } 1 \le p \le \infty. \tag{17}$$

Moreover, for all $1 \le p \le \infty$,

$$\nu_p \le \varrho(|A^{-1}|E) \le \alpha_p \nu_p, \text{ where } \alpha_p := n^{\min(1/p,1-1/p)}.$$

That means the new lower bound (17) is the same as (14) for $p \in \{1, \infty\}$, and better than (14) for all other values of p. Moreover, for $p = 2$ the bound (17) may be better up to a factor $\sqrt{n}$. This factor is achieved, for example, for all Hadamard matrices H by setting $A = n^{-1/2}H$, $E = |A|$. In this case $\varrho(|A^{-1}|\,|A|) = n \le \sqrt{n} \cdot \nu_2 \le \sqrt{n} \cdot \|A^{-1}\|_2 \|\,|A|\,\|_2 = n$ implies $\nu_2 = \sqrt{n} = n^{-1/2} \cdot \varrho(|A^{-1}|\,|A|)$. Is ν_2 always a global minimum of the ν_p?

Finally, the computation of ν_2 also yields a method to calculate an upper bound of $\underline{\omega}(A,E)$. This is done by calculating some $\widetilde{E}$ with $\det(A+\widetilde{E}) = 0$. Then $|\widetilde{E}| \le \beta E$ implies $\underline{\omega}(A,E) \le \beta$.

By (10) and (11) we have

$$\nu_2^{1/2} = \inf_{D \in \mathcal{D}_{2n}} \|D^{-1} \begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix} D\|_2. \tag{18}$$

Equality is seen as in the remark after the proof of Theorem 2.1: For $\|A^{-1}\| \ne \|E\|$ choose suitable $D_1 := \alpha I, D_2 := I$, block diagonal $D := \mathrm{diag}(D_1, D_2)$ and observe $\|D^{-1} \begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix} D\| = \max(\alpha\|A^{-1}\|,$ $\alpha^{-1}\|E\|)$. This proves $\|A^{-1}\| = \|E\|$ in optimality. For the moment assume the infimum is a minimum and A and E are scaled such that $\nu_2^{1/2} = \| \begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix} \|_2$. By (10) and the infimum process in the original definition (16) this implies $\|A^{-1}\|_2 = \|E\|_2 = \nu_2^{1/2} =: r$. Denote singular vectors of A^{-1} and E to r by $\|u\|_2 = \|v\|_2 = \|x\|_2 = \|y\|_2 = 1$ and $Ev = ru$, $A^{-1}y = rx$. Then

$$\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & v \end{pmatrix} = r \begin{pmatrix} 0 & u \\ x & 0 \end{pmatrix},$$

such that $\begin{pmatrix} y & 0 \\ 0 & v \end{pmatrix}$ and $\begin{pmatrix} 0 & u \\ x & 0 \end{pmatrix}$ span a 2-dimensional right and left singular vector space of $\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}$ to r. Suppose the multiplicity of the largest singular value r of $\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}$ is equal to 2. Then by [10] the minimization property of ν_2 implies that for $1 \le i \le n$ the i-th rows of $\begin{pmatrix} y & 0 \\ 0 & v \end{pmatrix}$ and $\begin{pmatrix} 0 & u \\ x & 0 \end{pmatrix}$ do have the same 2-norm. This means $|y| = |u|$ and $|v| = |x|$ or, for some signature matrices $|S_1| = |S_2| = I$, we have $y = S_1 u$ and $v = S_2 x$. Then

$$A^{-1}S_1ES_2x = A^{-1}S_1Ev = rA^{-1}S_1u = rA^{-1}y = r^2x,$$

and therefore

$$\det(r^2I - A^{-1}S_1ES_2) = 0 = \det(A - r^{-2}S_1ES_2).$$

For $\widetilde{E} := -r^{-2}S_1ES_2$ this means

$$\det(A + \widetilde{E}) = 0 \quad \text{and} \quad |\widetilde{E}| \le r^{-2} \cdot E = \nu_2^{-1}E.$$

In turn, if the multiplicity of the largest singular value of the minimizing matrix in (18) is two, then that of A^{-1} and E is one and the infimum in (18) is a minimum.

The minimization in (18) may be performed following [10] by minimizing the convex function $\|e^{-D}\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}e^D\|_2$. Suppose $\alpha = \|\begin{pmatrix} 0 & \widehat{E} \\ \widehat{A}^{-1} & 0 \end{pmatrix}\|_2 = \nu_2^{1/2}$ for $\widehat{E}$ and $\widehat{A}^{-1}$ being diagonally scaled E and A^{-1}, respectively. Then (18) and (17) imply

$$\alpha^{-2} \le \underline{\omega}(A, E). \tag{19}$$

Now take singular vectors u, v, x, y of E and A^{-1} as above and set $S_1 := \text{diag}(u \circ y)$, $S_2 := \text{diag}(v \circ x)$ with $\circ$ denoting the entrywise (Hadamard) product. If entries in u, v, x, y are zero, some heuristic may be applied to choose the corresponding diagonal elements of S_1, S_2 in $\{-1, 1\}$. For such S_1, S_2 define

$$\beta := \max\{|\lambda| : \ \lambda \textit{ real} \text{ eigenvalue of } A^{-1}S_1ES_2\},$$

with $\beta := 0$ if $A^{-1}S_1ES_2$ does not have a real eigenvalue. If $\beta \ne 0$, then

$$\det(s\beta I - A^{-1}S_1ES_2) = 0 = \det(A - \beta^{-1}sS_1ES_2)$$

for $s \in \{-1, +1\}$ with $|\beta^{-1}sS_1ES_2| = \beta^{-1}E$. By the definition (13) and (19) it follows

$$\alpha^{-2} \le \underline{\omega}(A, E) \le \beta^{-1}. \tag{20}$$

This process might be repeated for other signature matrices S_1, S_2, each pair producing a valid upper bound for $\underline{\omega}(A, E)$.

In a practical application computation of ν_2 to high accuracy is costly, although it can be solved by a convex minimization problem. Therefore it may be advisable to perform only a few minimization steps.

5. Numerical results. For different dimensions n we defined 100 random matrices $A \in M_n(\mathbb{R})$ with entries chosen randomly and uniformly distributed within $[-1, 1]$. For each matrix we computed bounds $[\underline{r}, \overline{r}]$ for $\omega(A, |A|)$ as in (20), that is with respect to relative perturbations of A. The following table displays the median and maximum relative error $(\overline{r} - \underline{r})/(\underline{r} + \overline{r})$.

	relative error	
n	median	maximum
10	$3.0 \cdot 10^{-3}$	$4.4 \cdot 10^{-2}$
20	$6.4 \cdot 10^{-3}$	$8.0 \cdot 10^{-2}$
50	$1.9 \cdot 10^{-6}$	$2.7 \cdot 10^{-2}$

In a number of cases the exact value of $\underline{\omega}(A, |A|)$ was enclosed fairly accurately within the bounds $[\underline{r}, \overline{r}]$. This seems remarkable for this NP-hard problem and the dimensions in use. The surprisingly good median value for $n = 50$ seems to have occurred accidentally.

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