Inclusion of zeros of nowhere differentiable n-dimensional functions *

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Abstract

A method is described for calculating verified error bounds for zeros of a system of nonlinear equations $f(x) = 0$ with continuous $f : \mathbb{R}^n \to \mathbb{R}^n$. We do not require existence of partial derivatives of f, and the function f may even be known only up to some finite precision $\varepsilon > 0$.

An inclusion may contain infinitely many zeros of f. An example of a continuous but nowhere differentiable function is given.

1 Notations

$\mathbb{P}S$ denotes the power set over some set S. For example, an element of $\mathbb{P}\mathbb{R}^{n\times n}$ is a set of matrices.

If not stated explicitly otherwise, all operations in use are power set operations. As usual, those are defined elementwise, i.e.

$$A \in \mathbb{P}S,\ B \in \mathbb{P}T \quad \Rightarrow \quad A \circ B := \{a \circ b \mid a \in A,\ b \in B\}$$

for suitable A, B, S, T and some operation $\circ$.

To allow application on the computer, we will replace power set operations by interval operations. The inclusion monotonicity of interval operations then yields computable inclusions.

*published in Reliable Computing, 3(1):5-16, 1997

2 Inclusion functions

Let $f : D \subseteq \mathrm{I\!R}^n \;\rightarrow\; \mathrm{I\!R}^n$ be a continuous function, and let

$$F : D \;\rightarrow\; \mathrm{I\!P I\!R}^n \quad \text{satisfy} \quad x \in D \;\Rightarrow\; f(x) \in F(x). \tag{1}$$

The function F can be thought of a computable error bound for f. For example, Stirling's formula yields (cf. [1])

$$\begin{aligned} & f(x) := \Gamma(x) \quad \text{and} \\ & F(x) := \sqrt{2\pi} \cdot (x-1)^{x-\frac{1}{2}} \cdot e^{-x+1+\frac{[0,1]}{12(x-1)}}. \end{aligned} \tag{2}$$

f and F satisfy condition (2.1) for $D = \{x \in \mathrm{I\!R} \,|\, x > 0\}$, where the expression for F can be evaluated using interval arithmetic and is executable on digital computers.

Second, we need a local expansion of f w.r.t. some $\widetilde{x} \in D$. That means we suppose existence of some function $s_f \;:\; D \times \mathrm{I\!P}D \;\rightarrow\; \mathrm{I\!P I\!R}^{n \times n}$ with

$$\widetilde{x} \in D,\; X \in \mathrm{I\!P}D,\; x \in X \;\Rightarrow\; f(x) \in f(\widetilde{x}) + s_f(\widetilde{x}, X) \cdot (x - \widetilde{x}). \tag{3}$$

We want to stress that we do not assume $\widetilde{x} \in X$. There are many possibilities to compute such a function s_f, for example by an automized slope computation [10], [4], [5], [9], [2]. This process is very similar to automatic differentiation [7], [3]. It is performed in such a way that (2.3) is preserved in every operation. We have no further assumptions on s_f such as continuity. The set of matrices $s_f(\widetilde{x}, X)$ need not even be connected.

A slope can be computed for $\widetilde{x} \in D$ and some $X \subseteq D$ using triples $(f_c, f_r, f_s) \in \mathrm{I\!P I\!R}^3$ satysfying

$$f(\widetilde{x}) \in f_c, \quad f(X) \subseteq f_r, \quad \forall\, x \in X \;:\; f(x) \in f(\widetilde{x}) + f_s \cdot (x - \widetilde{x}). \tag{4}$$

Then operations $+, -, \cdot, /$, transcendental functions and so forth are defined for those tripels. For details see [5] and for some improvements [9]. The initialisation consists of

$$\begin{aligned} & f(x) \equiv c = \text{const} && \Rightarrow \quad f_c = c,\; f_r = c,\; f_s = 0 \quad \text{and} \\ & f(x) = x && \Rightarrow \quad f_c = \widetilde{x},\; f_r = X,\; f_s = 1. \end{aligned}$$

This definition satisfies (2.4). The interesting fact is that if f is given only up to some finite precision like in (2.2), the process of automatic slope evaluation still applies. A set constant $C \in \mathrm{I\!P I\!R}$ is represented by

$$f_c = C, \quad f_r = C \quad \text{and} \quad f_s = 0.$$

As an example, we demonstrate the slope evaluation of $g(x) := e^{-x+1+\frac{[0,1]}{12(x-1)}}$ for $\tilde{x} = 4$ and $X = [2.6,\ 2.8]$. We have

$$\begin{array}{llll}
\alpha(x) := [0,\ 1] & \Rightarrow \alpha_c = [0,\ 1], & \alpha_r = [0, 1], & \alpha_s = 0 \\
\beta(x) := 12 & \Rightarrow \beta_c = 12, & \beta_r = 12, & \beta_s = 0 \\
\gamma(x) := x & \Rightarrow \gamma_c = 4, & \gamma_r = [2.6,\ 2.8], & \gamma_s = 1 \\
\delta(x) = \gamma(x)\text{-}1 & \Rightarrow \delta_c = 3, & \delta_r = [1.6,\ 1.8], & \delta_s = 1 \\
\varepsilon(x) = \beta(x) \cdot \delta(x) & \Rightarrow \varepsilon_c = 36, & \varepsilon_r = [19.2,\ 21.6], & \varepsilon_s = 12 \\
\zeta(x) = \alpha(x)/\varepsilon(x) & \Rightarrow \zeta_c = [0,\ 0.0278], & \zeta_r = [0,\ 0.0521], & \zeta_s = [\text{-}0.0174,\ 0] \\
\eta(x) = \zeta(x)\text{-}\delta(x) & \Rightarrow \eta_c = [\text{-}3, \text{-}2.9722], & \eta_r = [\text{-}1.8, \text{-}1.5479], & \eta_s = [\text{-}1.0174, \text{-}1] \\
\vartheta(x) = e^{\eta(x)} & \Rightarrow \vartheta_c = [0.0498,\ 0.0512], & \vartheta_r = [0.1653,\ 0.2127], & \vartheta_s = [\text{-}0.1154, \text{-}0.0963]
\end{array}$$

using

$$\begin{aligned}
(f \cdot g)_s &= (f_s \cdot g_r + f_c \cdot g_s) \cap (f_r \cdot g_s + f_s \cdot g_c) \\
(f/g)_s &= (f_s - h_c \cdot g_s) \,/\, g_r \cap (f_s - h_r \cdot g_s) \,/\, g_c \quad \text{with} \quad h := f/g.
\end{aligned}$$

(cf. [9]). Furthermore, the improved way for calculating slopes of locally convex functions [9] is used to calculate ϑ_s. Continuing in this way we obtain

$$\Gamma(x) \in [1.11,\ 2.07] \quad \text{for} \quad x \in X = [2.6,\ 2.8],$$

a valid inclusion obtained by the somethat crude expansion (2.2). As a result of this chapter we want to stress that an inclusion function F satisfying (2.1) and an expansion function s_f satisfying (2.3) can be computed for many functions f including those which are only known up to a finite error margin. Basically, f has to be given by a computer program (for a precise definition using the concept of algebras cf. [5]). The method is based on an expansion of f within X w.r.t. some $\tilde{x}$, where $\tilde{x}$ need not be an element of X.

3 Inclusion theorems

With these assumptions, an inclusion theorem can be proved along the lines of [9]. Let continuous $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ and functions F and s_f satisfying (2.1) and (2.3), respectively be given. Furthermore, let $X \subseteq D$ be nonempty, convex and compact, $\tilde{x} \in D$ and $R \in \mathbb{R}^{n \times n}$. Recommended but not assumed are $\tilde{x}$ to be an approximation of a zero of f, X a small region including $\widetilde{x}$, and R to be an approximate inverse of some matrix within $s_f(\widetilde{x}, X)$. The function $g : D \rightarrow \mathrm{IR}^n$ defined by

$$g(x) := x - R \cdot f(x) \tag{5}$$

is continuous, and for every $x \in X$ (2.3) yields

$$g(x) \in x - R \cdot \{\, f(\widetilde{x}) + s_f(\widetilde{x},\, X) \cdot (x - \widetilde{x})\,\}.$$

That means, there exists some matrix $M \in s_f(\widetilde{x}, X)$ with

$$\begin{aligned} g(x) &= x - R \cdot \{\, f(\widetilde{x}) + M \cdot (x - \widetilde{x})\,\} = \widetilde{x} - R \cdot f(\widetilde{x}) + \{\, I - R \cdot M\,\} \cdot (x - \widetilde{x}) \\ &\in \widetilde{x} - R \cdot F(\widetilde{x}) + \{\, I - R \cdot s_f(\widetilde{x}, X)\,\} \cdot (X - \widetilde{x}). \end{aligned} \tag{6}$$

If the r.h.s. of (3.2) is included in X, this implies $g(X) \subseteq X$, and Brouwer's Fixed Point Theorem yields existence of some $\widehat{x} \in X$ with $g(\widehat{x}) = \widehat{x}$ and $R \cdot f(\widehat{x}) = 0$. The same is true when assuming an interval version of the r.h.s. of (3.2) to be included in X.

$\widehat{x}$ is a zero of f if R is regular. This can be verified on the computer by assuming inclusion in the interior $\mathring{X}$ of X rather than in X (cf. [8], [9]). This proves the following theorem.

Theorem 3.1. Let $f : D \subseteq \mathrm{IR}^n \rightarrow \mathrm{IR}^n$ be continuous, F and s_f be given according to (2.1) and (2.3), $\emptyset \neq X \subseteq D$ compact and convex, $\widetilde{x} \in D$ and $R \in \mathrm{IR}^{n \times n}$. If

$$\widetilde{x} - R \cdot F(\widetilde{x}) + \{I - R \cdot s_f(\widetilde{x}, X)\,\} \cdot (X - \widetilde{x}) \subseteq \mathring{X},$$

then there exists some $\widehat{x} \in X$ with $f(\widehat{x}) = 0$.

As has been pointed out already in [8], it is superior to calculate an inclusion of $\widehat{x} - \widetilde{x}$ rather than including $\widehat{x}$ itself. This observation (cf. also [9] and the references cited over there for a detailed discussion) is applied in our following example.

4 Application to nowhere differentiable functions

Weierstraß [11] defines an everywhere continuous but nowhere differentiable function from IR to IR as follows. Let

$0 < \alpha < 1, \quad b \in \mathbb{N}$ even with $\quad \alpha \cdot b > 1 + \frac{3}{2}\pi \quad$ and

$$w_n(x) := \sum_{\nu=1}^{n} \alpha^{\nu} \cdot \sin(b^{\nu} \cdot \pi x). \tag{7}$$

Then

$$w : \mathbb{R} \to \mathbb{R} \text{ with } w(x) := \lim_{n\to\infty} w_n(x)$$

is well-defined, everywhere continuous but nowhere differentiable. The idea goes back to Bolzano. For fixed $k \in \mathbb{N}$ we have

$$\left| \sum_{\nu=k+1}^{\infty} \alpha^{\nu} \cdot \sin(b^{\nu} \cdot \pi x) \right| \leq \sum_{\nu=k+1}^{\infty} \alpha^{\nu} = \frac{\alpha^{k+1}}{1-\alpha}.$$

Therefore, a function $W : \mathbb{R} \to \mathbb{IPR}$ satisfying (2.1) for w, W instead of f, F can be defined by

$$W(x) := \sum_{\nu=1}^{k} \alpha^{\nu} \cdot \sin(b^{\nu} \cdot \pi x) + [-c, c] \quad \text{with} \quad c := \frac{\alpha^{k+1}}{1-\alpha}. \tag{8}$$

A corresponding expansion s_W can be computed by an automized process as described in Chapter 2.

The definition of the function f implies some "fractalness": arbitrary zooming at any point always decovers new peaks of w. Therefore, one may assume that zeros of w come in clusters. This is true, and is made precise in the following theorem.

Theorem 4.1. Let $n \in \mathbb{N}$ and $w_n(\widehat{x}) = 0$ for some $\widehat{x} \in \mathbb{R}$. Then infinitely many zeros x of w satisfy

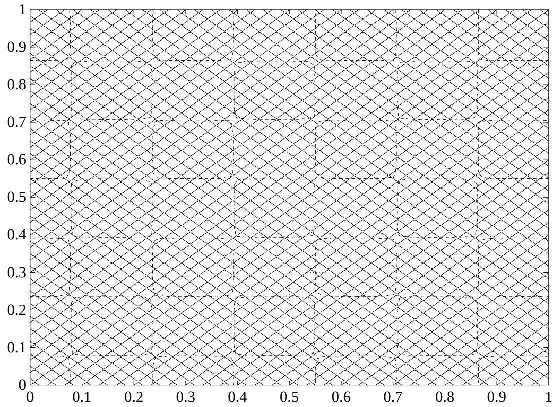
$$|x - \widehat{x}| \leq b^{-n} \tag{9}$$

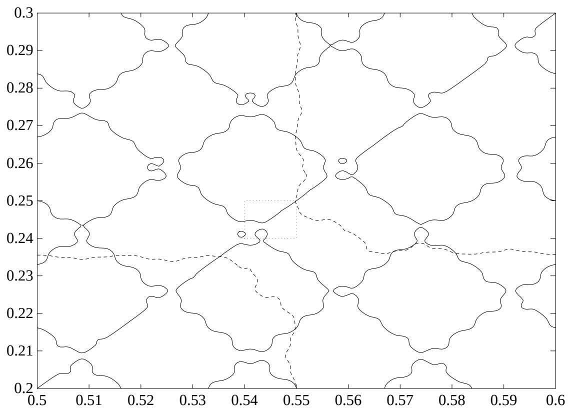
The proof is technical and put into the Appendix. Note that every integer x is a zero of w_1.

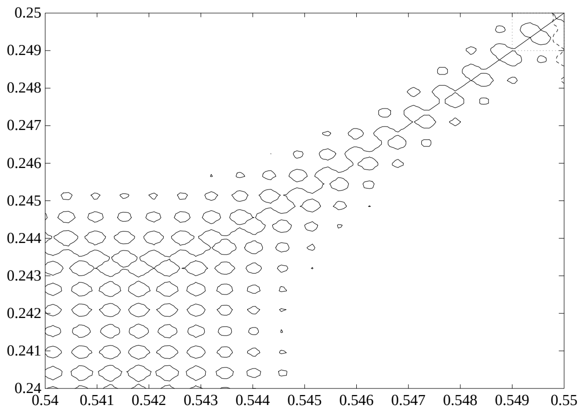
Computation of the zeros of one-dimensional functions could be performed by checking the sign of the values at the endpoints. To make it more interesting we define

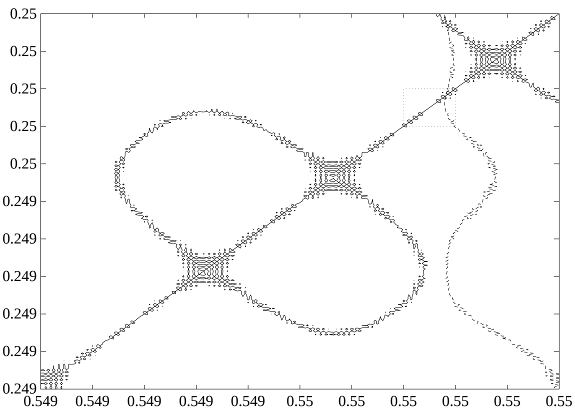
$$\begin{aligned}
&f : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{by} \quad f(x,y) := (f_1(x,y),\ f_2(x,y))^T \quad \text{with} \\
&f_1(x,y) := e^{w(x)} - w(y) - 1 \\
&f_2(x,y) := w(x)^2 + w(y)^2 - \cos(20x) \cdot \cos(20y)/\sqrt{2}.
\end{aligned}$$

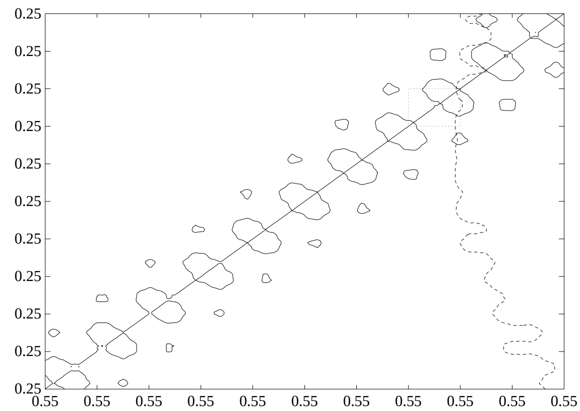
In the following we display plots of zeros of f_1 and f_2 using the contour-function in MATLAB [6]. The dashed lines are the zero lines of f_1, the full lines the zero lines of f_2. We start with the box $[0,1] \times [0,1]$ and zoom each plot by a factor of 10, where the range of the succeeding plot is depicted by the small dotted box. While zooming, the contour-function displays more and more details of the zero lines of f_1 and f_2.











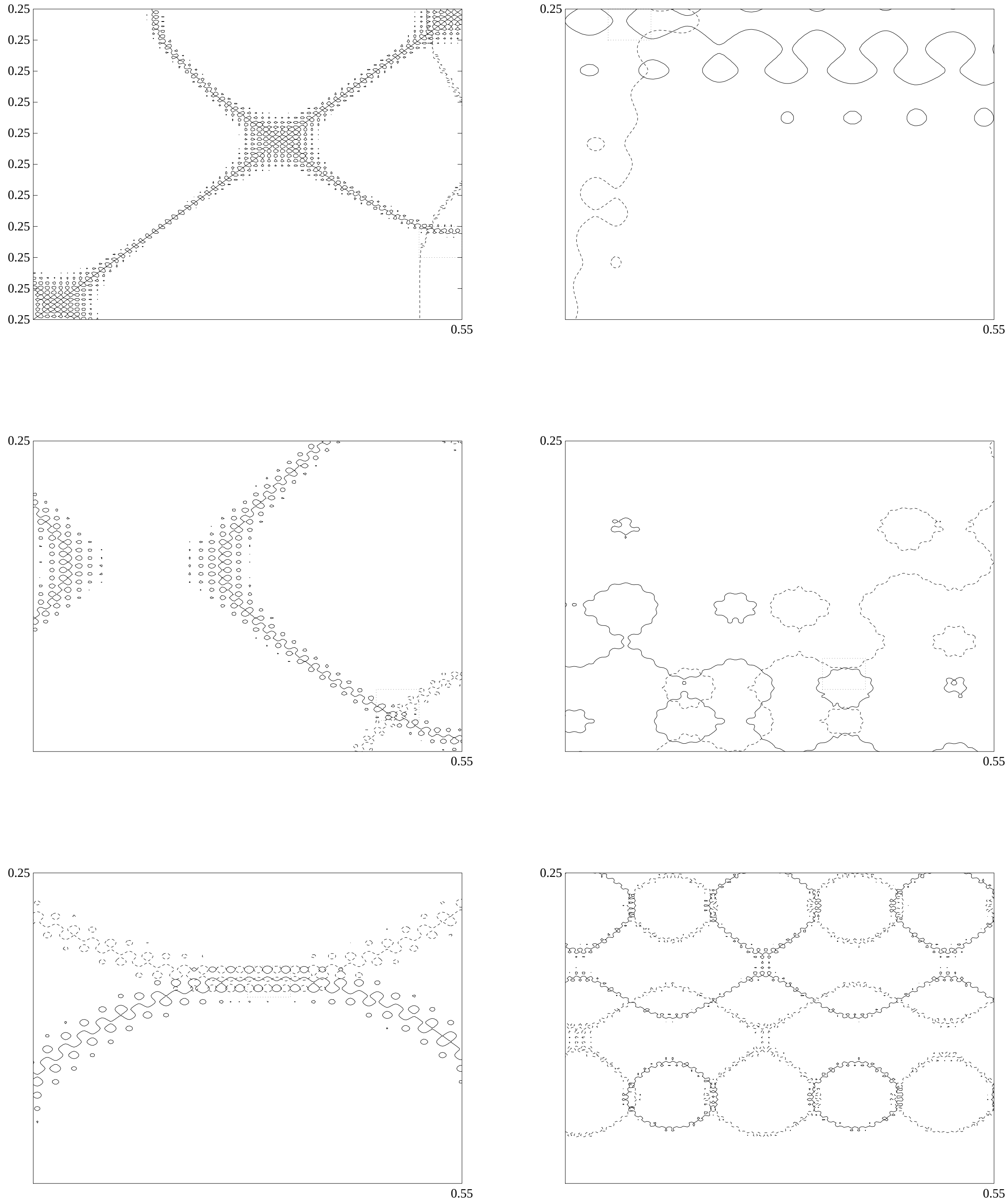


Table 4.2. Plots of zeros of f_1 and f_2.

The following table lists inclusions of the visible zeros in the box $[0.5, 0.6] \times [0.2, 0.3]$ by our method. We use IEEE 754 double precision equivalent to 16 to 17 decimals.

$$\begin{array}{ll}
\widehat{x}_1 \in 0.5021204500^{7}_{5} & \widehat{y}_1 \in 0.23546280035^{9}_{7} \\
\widehat{x}_2 \in 0.514782868861^{5}_{0} & \widehat{y}_2 \in 0.23544326806^{9}_{6} \\
\widehat{x}_3 \in 0.5350276990^{2}_{0} & \widehat{y}_3 \in 0.2350324011^{8}_{6} \\
\widehat{x}_4 \in 0.549777160^{6}_{4} & \widehat{y}_4 \in 0.216443827^{4}_{1} \\
\widehat{x}_5 \in 0.549778^{5}_{1} & \widehat{y}_5 \in 0.2004^{305}_{294} \\
\widehat{x}_6 \in 0.549778878^{8}_{5} & \widehat{y}_6 \in 0.299984318^{6}_{2} \\
\widehat{x}_7 \in 0.549778889658^{6}_{1} & \widehat{y}_7 \in 0.28331574618^{8}_{4} \\
\widehat{x}_8 \in 0.549779007^{4}_{1} & \widehat{y}_8 \in 0.267040509^{4}_{0} \\
\widehat{x}_9 \in 0.549779247^{8}_{5} & \widehat{y}_9 \in 0.249779247^{8}_{5} \\
\widehat{x}_{10} \in 0.5705553818^{9}_{7} & \widehat{y}_{10} \in 0.2372325822^{5}_{3} \\
\widehat{x}_{11} \in 0.580277793^{3}_{1} & \widehat{y}_{11} \in 0.236374696^{6}_{3}
\end{array}$$

Table 4.3. Inclusion of visible zeros of f within $[0.5, 0.6] \times [0.2, 0.3]$

However, there are also zeros for which the contour plot of the zero lines of f_1 and f_2 does not give a hint to their existence. For example,

$$\widehat{x}_{12} \in 0.5487^{50009}_{49998} \qquad \widehat{y}_{12} \in 0.2343841^{9}_{7}$$

contains a zero, which is depicted by the small circle in the following contour plot over the box $[0.5, 0.6] \times [0.2, 0.3]$.

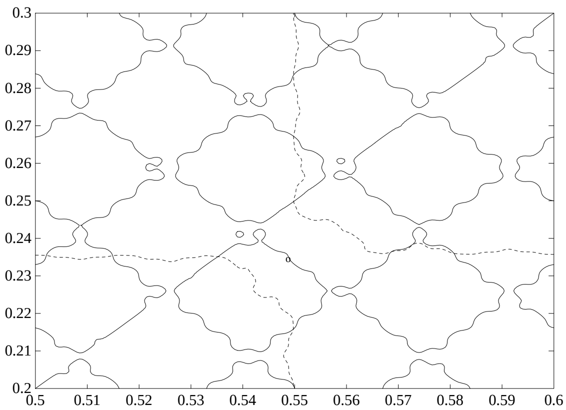


Table 4.4. Box $[0.5, 0.6] \times [0.2, 0.3]$ with extra zero of f.

The zeros of the last plot in Table 4.2 can be included as well. Sample inclusions are listed in the following table.

$\widehat{x}_1 \in 0.5497791865^{15}_{09}$	$\widehat{y}_1 \in 0.2497729126^{23}_{17}$
$\widehat{x}_2 \in 0.5497791865537^{4}_{2}$	$\widehat{y}_2 \in 0.2497729126200^{8}_{0}$
$\widehat{x}_3 \in 0.5497791865554^{9}_{7}$	$\widehat{y}_3 \in 0.2497729126^{201}_{199}$
$\widehat{x}_4 \in 0.5497791865808^{8}_{3}$	$\widehat{y}_4 \in 0.249772912620^{40}_{34}$
$\widehat{x}_5 \in 0.5497791865969^{6}_{1}$	$\widehat{y}_5 \in 0.249772912620^{42}_{36}$
$\widehat{x}_6 \in 0.54977918651^{5}_{0}$	$\widehat{y}_6 \in 0.24977291263^{88}_{39}$
$\widehat{x}_7 \in 0.549779186537^{32}_{27}$	$\widehat{y}_7 \in 0.249772912636^{13}_{06}$
$\widehat{x}_8 \in 0.549779186554^{83}_{77}$	$\widehat{y}_8 \in 0.249772912636^{16}_{08}$
$\widehat{x}_9 \in 0.5497791865808^{8}_{3}$	$\widehat{y}_9 \in 0.2497729126357^{8}_{2}$
$\widehat{x}_{10} \in 0.5497791865969^{6}_{1}$	$\widehat{y}_{10} \in 0.2497729126357^{6}_{0}$
$\widehat{x}_{11} \in 0.5497791865^{142}_{099}$	$\widehat{y}_{11} \in 0.24977291265^{75}_{39}$
$\widehat{x}_{12} \in 0.549779186537^{34}_{26}$	$\widehat{y}_{12} \in 0.249772912655^{6}_{4}$
$\widehat{x}_{13} \in 0.549779186554^{83}_{77}$	$\widehat{y}_{13} \in 0.249772912656^{2}_{0}$
$\widehat{x}_{14} \in 0.54977918658085^{6}_{4}$	$\widehat{y}_{14} \in 0.24977291265611^{4}_{2}$
$\widehat{x}_{15} \in 0.5497791865969^{6}_{1}$	$\widehat{y}_{15} \in 0.2497729126561^{6}_{0}$

Table 4.5. Inclusion of zeros of f within $[0.549779186^{6}_{5}] \times [0.249772912^{7}_{6}]$.

The zeros of the continuous but nowhere differentiable function f in this box differ only in the 11th place after the decimal point. Inclusions are still possible demonstrating the capabilities of the method.

Using theoretical considerations like Theorem 4.1 computation of bounds of zeros can be derived from bounds for zeros of the function obtained by using the finite sum (4.1) instead of $w(x)$. However, we wish to emphasize that the described method can be used in a fully automized computer program using only the inclusion function $W(x)$.

Appendix. The proof of Theorem 4.1 needs some preliminary lemmata. In addition to (4.1) and (4.2) we set

$$p_\nu(x) := \alpha^\nu \cdot \sin(b^\nu \cdot \pi x) \quad \text{such that} \quad w_n(x) = \sum_{\nu=1}^{n} p_\nu(x). \tag{1}$$

Note that $0 < \alpha < 1$, $\alpha b > 1 + \frac{3}{2}\pi$ and $b \in \mathbb{N}$ even implies $b \geq 6$. p_n has the period $2 \cdot b^{-n}$, that is

$$p_n(x) = p_n(x + 2 \cdot b^{-n}) \quad \text{for all} \quad n \in \mathbb{N}. \tag{2}$$

Furthemore, $w_n(x)$ is differentiable, and for all $x \in \mathbb{R}$,

$$| \, w_n'(x) \, | \; = | \; \pi \cdot \sum_{\nu=1}^{n} (\alpha b)^{\nu} \cdot \cos(b^{\nu} \cdot \pi x) \, | \; \leq \; \pi \cdot \tfrac{(\alpha b)^{n+1}-1}{\alpha b-1} \; < \; \tfrac{2}{3} \cdot (\alpha b)^{n+1}. \tag{3}$$

Lemma 1. Let $n \in \mathbb{N}$, $\widehat{x} \in \mathbb{R}$ and $w_n(\widehat{x}) = 0$. Then there are $\widetilde{x}_1, \widetilde{x}_2 \in \mathbb{R}$ with $w_{n+1}(\widetilde{x}_i) = 0$, $i = 1, 2$ and

$$\widetilde{x}_1 \; \leq \; \widehat{x} \; < \; \widehat{x}_2, \quad \tfrac{2}{5} \cdot b^{-n-1} \; < \; \widetilde{x}_2 - \widehat{x} \; \leq \; \widetilde{x}_2 - \widetilde{x}_1 \; < \; \tfrac{11}{5} \cdot b^{-n-1} \tag{4}$$

and

$$|\widetilde{x}_1 - \widehat{x}| \; \leq \; \tfrac{2}{3} \cdot b^{-n-1}, \quad |\widetilde{x}_2 - \widehat{x}| \; \leq \; \tfrac{7}{5} \cdot b^{-n-1} \tag{5}$$

Proof. Set $k := \lfloor b^{n+1} \cdot \widehat{x} \rfloor \in \mathbb{N}$. Without loss of generality we may assume $p_{n+1}(\widehat{x}) \geq 0$ with

$$k \cdot b^{-n-1} \; \leq \; \widehat{x} \; \leq \; (k + \tfrac{1}{2}) \cdot b^{-n-1}. \tag{6}$$

That means $\widehat{x}$ lies in the first quarter of the period of p_{n+1} and $0 \leq p_{n+1}(\widehat{x}) \leq \alpha^{n+1}$.

It is

$$w_{n+1}(\widehat{x}) = w_n(\widehat{x}) + p_{n+1}(\widehat{x}) = p_{n+1}(\widehat{x}) \geq 0, \tag{7}$$

and for $\varepsilon \in \mathbb{R}$ and suitable $\zeta \in \mathbb{R}$ holds

$$w_{n+1}(\widehat{x} + \varepsilon) = w_n(\widehat{x}) + w_n'(\zeta) \cdot \varepsilon + p_{n+1}(\widehat{x} + \varepsilon) = w_n'(\zeta) \cdot \varepsilon + p_{n+1}(\widehat{x} + \varepsilon). \tag{8}$$

Set $y := (k - \frac{1}{6}) \cdot b^{-n-1}$. Then (6) implies $0 \leq \widehat{x} - y \leq \frac{2}{3} \cdot b^{-n-1}$, (8), (2) and (3) yield

$$w_{n+1}(y) \; \leq \; \tfrac{2}{3}\alpha^{n+1} \cdot \tfrac{2}{3} + p_{n+1}(y) \; < \; \tfrac{4}{9} \cdot \alpha^{n+1} - \tfrac{1}{2} \cdot \alpha^{n+1} \; < \; 0.$$

Hence, (7) implies existence of some $\widetilde{x}_1$ with $w_{n+1}(\widetilde{x}_1) = 0$ and

$$(k - \tfrac{1}{6}) \cdot b^{-n-1} \; < \; \widetilde{x}_1 \; \leq \; \widehat{x}. \tag{9}$$

Together with (6) this yields the first part of (5). For $\varepsilon := \frac{2}{5} \cdot b^{-n-1}$, (8), (6) and (3) imply

$$\begin{aligned} w_{n+1}(\widehat{x} + \varepsilon) \; &\geq \; -\tfrac{2}{3} \cdot \alpha^{n+1} \cdot \tfrac{2}{5} + p_{n+1}(\widehat{x} + \varepsilon) \\ &> \; -\tfrac{4}{15} \cdot \alpha^{n+1} + \min\{ \sin(\pi x) \mid \tfrac{2}{5} \; \leq \; x \; \leq \; \tfrac{2}{5} + \tfrac{1}{2} \} \cdot \alpha^{n+1} \\ &> \; -\tfrac{4}{15} \cdot \alpha^{n+1} + \tfrac{3}{10} \cdot \alpha^{n+1} \; > \; 0. \end{aligned} \tag{10}$$

Furthemore, for $y := (k + \frac{7}{5}) \cdot b^{-n-1}$, (6) gives $0 \leq y - \widehat{x} \leq \frac{7}{5} \cdot b^{-n-1}$, and (8), (2) and (3) imply

$$w_{n+1}(y) \; \leq \; \tfrac{2}{3} \cdot \alpha^{n+1} \cdot \tfrac{7}{5} + \sin(\tfrac{7}{5}\pi) \cdot \alpha^{n+1} < 0.$$

Now, (10) implies existence of $\widetilde{x}_2$ with $w_{n+1}(\widetilde{x}_2) = 0$ and

$$\widehat{x} + \tfrac{2}{5} \cdot b^{-n-1} \; < \; \widetilde{x}_2 \; \leq \; \widehat{x} + \tfrac{7}{5} \cdot b^{-n-1}.$$

Together with (9) and (6) this proves lemma. ■

Lemma 2. Let $n \in \mathbb{N}$, $\widehat{x} \in \mathbb{R}$ and $w_n(\widehat{x}) = 0$. Then there is some $z \in \mathbb{R}$ with

$$f(z) = 0 \quad \text{and} \quad |z - \widehat{x}| \leq \tfrac{7}{8} \cdot b^{-n-1}. \tag{11}$$

Proof. According to (5) in Lemma 1 and $b \geq 6$, there is a sequence $x_\nu|_{\nu \geq 0}$ with $x_0 := \widehat{x}$ and

$$|\widehat{x} - x_\nu| \leq \tfrac{7}{10} \cdot b^{-n-1} \cdot \sum_{i=0}^{\infty} b^{-i} < \tfrac{7}{8} \cdot b^{-n-1}.$$

(x_ν) is bounded, and there is a convergent subsequence $x_{\nu_k} \to z$ with $|z - \widehat{x}| \leq \frac{7}{8} \cdot b^{-n-1}$. Definition (1) implies for all $x \in \mathbb{R}$

$$|\, w_\mu(x) - w(x) \,| = |\sum_{\nu=\mu+1}^{\infty} p_\nu(x) \,| \leq \alpha^{\mu+1} \cdot \sum_{\nu=0}^{\infty} \alpha^{-\nu} = \tfrac{\alpha^{\mu+1}}{1-\alpha}.$$

That means

$$\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \quad \forall k \geq k_0 : \quad |x_{\nu_k} - z| < \varepsilon \;\text{ and }\; |w(x_{\nu_k})| < \varepsilon.$$

The continuity of f implies $f(z) = 0$. The lemma is proved. ■

Lemma 3. Let $n \in \mathbb{N}, \widehat{x} \in \mathbb{R}$ and $w_n(\widehat{x}) = 0$ and some $1 \leq k \in \mathbb{N}$ be given. Then there are strictly increasing $x_i \in \mathbb{R}$, $0 \leq i \leq k$ with

$$w_{n+k}(x_i) = 0 \;\text{ and }\; |x_i - \widehat{x}| \leq 2 \cdot b^{-n-1} \quad \text{for } 0 \leq i \leq k, \text{ and} \tag{12}$$

$$\tfrac{2}{5} \cdot b^{-n-k} < x_{i+1} - x_i \qquad \text{for } 0 \leq i \leq k-1. \tag{13}$$

Proof. We prove by induction for $1 \leq k \in \mathbb{N}$,

$$|x_i - \widehat{x}| \leq \tfrac{7}{5} \cdot b^{-n-1} \cdot \sum_{\nu=0}^{k-1} b^{-\nu}. \tag{14}$$

Lemma 1 implies that (13) and (14) are true for $k = 1$. Suppose (13) and (14) are satisfied for $k \in \mathbb{N}$. By Lemma 1, for every x_i, $0 \leq i \leq k$ there exist $\underline{x_i}$, $\overline{x}_i$ with $w_{n+k+1}(\underline{x_i}) = w_{n+k+1}(\overline{x_i}) = 0$ and

$$\underline{x}_i \leq x_i < \overline{x}_i \;\text{ and }\; \tfrac{2}{5} \cdot b^{-n-1-k} < \overline{x}_i - \underline{x}_i < \tfrac{11}{5} \cdot b^{-n-1-k}. \tag{15}$$

We will show that the numbers $\underline{x}_0, \overline{x}_0, \overline{x}_1, \cdots, \overline{x}_k$ satisfy our assertions (12), (13) and (14). (15) and $b \geq 6$ imply for $0 \leq i \leq k$,

$$\begin{array}{rcl} \overline{x}_{i+1} - \overline{x}_i &=& \overline{x}_{i+1} - x_{i+1} + x_{i+1} - x_i - (\overline{x}_i - x_i) \\ &\geq& \frac{2}{5} \cdot b^{-n-1-k} + \frac{2}{5} \cdot b^{-n-k} - \frac{11}{5} \cdot b^{-n-k-1} \\ &>& (\frac{2}{5} + \frac{12}{5} - \frac{11}{5}) \cdot b^{-n-k-1} \\ &>& \frac{2}{5} \cdot b^{-n-k-1}. \end{array}$$

Together with (15) this demonstrates (13). With (14) we obtain

$$|\overline{x}_k - \widehat{x}| \leq |x_k - \widehat{x}| + |\overline{x}_k - x_k| \leq \tfrac{7}{5} \cdot b^{-n-1} \cdot \sum_{\nu=0}^{k-1} b^{-\nu} + \tfrac{7}{5} \cdot b^{-n-1-k}.$$

This verifies (14), and using $b \geq 6$ and

$$\sum_{\nu=0}^{\infty} b^{\nu} = \frac{6}{5}$$

proves the lemma. ■

Proof of Theorem 4.1. Let $2 \leq k \in \mathbb{N}$ be given. According to Lemma 3, there are strictly increasing $x_i \in \mathbb{R}$, $0 \leq i \leq k$ with

$$w_{n+k}(x_i) = 0, \quad |x_i - \widehat{x}| \leq 2 \cdot b^{-n-1} \quad \text{and} \quad x_{i+1} - x_i > \tfrac{2}{5} \cdot b^{-n-k}. \tag{16}$$

According to Lemma 2, there exist $z_i \in \mathbb{R}$, $0 \leq i \leq k$ with

$$w(z_i) = 0 \quad \text{and} \quad |z_i - x_i| \leq \tfrac{7}{8} \cdot b^{-n-k-1}. \tag{17}$$

Using $b \geq 6$ together with (16) and (17) yields for $1 \leq i \leq k-1$,

$$\begin{aligned} |z_{i+1} - z_{i-1}| &\geq |x_{i+1} - x_{i-1}| - |x_{i+1} - z_{i+1}| - |z_{i-1} - x_{i-1}| \\ &\geq 2 \cdot \tfrac{2}{5} \cdot b^{-n-k} - 2 \cdot \tfrac{7}{8} \cdot b^{-n-k-1} \\ &> 2 \cdot b^{-n-k} \cdot \left(\tfrac{2}{5} - \tfrac{7}{8} \cdot \tfrac{1}{6}\right) > 0. \end{aligned} \tag{18}$$

Moreover,

$$|z_i - \widehat{x}| \leq |z_i - x_i| + |x_i - \widehat{x}| \leq \tfrac{7}{8} \cdot b^{-n-k-1} + 2 \cdot b^{-n-1} < b^{-n}. \tag{19}$$

In other words, there are $\lfloor k/2 \rfloor$ zeros of $w(x)$, which are distinct by (18), satisfying (4.3). ■

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